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Harmonic oscillator Floquet states in the Bargmann–Segal space

A Palma[1,5], **V Leon**[2] **and R Lefebvre**[3,4]

[1] Instituto Nacional de Astrofísica, Optica y Electrónica (INAOE), Apdo Postal 51 y 216, Puebla, Pue. 72000, Mexico
[2] Instituto de Física, BUAP, Apdo Postal J-48, Puebla, Pue. 72570, Mexico
[3] Laboratoire de Photophysique Moléculaire du CNRS, Bâtiment 213, Université Paris-Sud, 91405 Orsay, France
[4] UFR de Physique Fondamentale et Appliquée, Université Pierre et Marie Curie 75231 Paris, France

E-mail: palma@sirio.ifuap.buap.mx

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Abstract
The Floquet quasi-energies and eigenfunctions for the harmonic oscillator interacting with a monochromatic electric field are obtained by using the so-called Bargmann–Segal space. The Schrödinger second-order differential equation in configuration space is transformed into a linear first-order equation in such a space, which is easily solved by means of an auxiliary system (called the Lagrange system) of ordinary differential equations. This method compares favourably with others previously used.

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1. Introduction

The second-order Schrödinger differential equation represents one of the greatest achievements of quantum mechanics, so that any method to solve it in a simple way should be considered as an important step in the theory of Schrödinger solutions. In this context, we refer to the Bargmann–Segal (BS) [1] transformation, which is a useful method often applied to solve some problems [2, 3], by using well known properties of the boson operators. Applying this transformation to other systems, such as the harmonic oscillator interacting with an external field, should be the next step to consider. Although these kinds of problems have been solved before via traditional analytical methods [4], operator algebra techniques [5] and the path integration method [6], it is important to look for those which lead more easily to the exact solutions.

[5] On sabbatical leave from Instituto de Física (BUAP).

The organization of the paper is as follows: in section 2 we start with the original heuristic proposal of Fock [7], masterfully discussed by Dirac [8] and several years later rigorously justified by Bargmann [9] and Segal [10]. In the BS representation it is well known [1, 8] that the harmonic oscillator wavefunctions are monomials, in clear contrast with the Hermite polynomials multiplied by a Gaussian function which are the solutions in the configuration space; this simplification plays a key role in our work. Although for the quantum forced harmonic oscillator the path integration method [11] has been successfully used and even a relationship of this method has been found with the BS variables [12], in section 3 we will show how the use of a technique developed by Lagrange leads to the Floquet eigenfunctions and quasi-energies in a straightforward way. Finally, in section 4 some conclusions are given.

2. The Bargmann–Segal transformation

A long time ago Fock [7] proposed an operator solution to the commutator relation

$$[a, a^\dagger] = 1 \tag{1}$$

where a and $a^\dagger$ are the annihilation and creation operators associated with the harmonic oscillator. This solution can be formulated in the field of complex numbers if we consider

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}, z\right] = 1 \tag{2}$$

where z is a complex variable and thus we have the transformation rule

$$a^\dagger \to z \qquad a \to \frac{\mathrm{d}}{\mathrm{d}z}. \tag{3}$$

The Hamiltonians that are written in the second quantization language can be transformed to the BS space by applying this rule; a trivial example is the harmonic oscillator itself:

$$H = a^\dagger a + \frac{1}{2} \quad \longrightarrow \quad H = z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}. \tag{4}$$

The solution to the eigenvalue equation via this transformed Hamiltonian is easily found to be

$$\chi_n(z) = z^n \qquad E_n = n + \tfrac{1}{2} \quad (n \text{ positive integer}). \tag{5}$$

The quantization in this case (n positive integer) does not follow from a boundary condition, as in the configuration space, but from the analyticity of the complex variable function. The rigorous mathematical justification of the transformation rule (equation (3)) was given in the paper by Bargmann [9], Segal [10] (in this reference it is called the holomorphic functional representation) and Cook [13], where the general problem of changing space was dealt with by means of a linear transformation between two Hilbert spaces: the usual configuration space and the entire space of complex functions [14].

3. The Floquet theorem and the Lagrange system

Let us consider a particle in a harmonic potential interacting with an external monochromatic field. By introducing the boson operators, the time-dependent Schrödinger equation in the configuration space is

$$H\Psi = \mathrm{i}\frac{\partial \Psi}{\partial t} \tag{6}$$

where the Hamiltonian is given by

$$H = \omega_0(2a^\dagger a + 1) + \frac{\lambda}{\sqrt{2\omega_0}}(a + a^\dagger)\cos\omega t. \tag{7}$$

The last term represents the interaction between the particle and the semiclassical electric field of frequency ω, and ω_0 is the frequency associated with the oscillator. It is well known [15] that there exist solutions of equation (6) of the form

$$\Psi_F^n(x,t) = \exp\left[-\mathrm{i}E_F^n t\right]\,\Phi_F^n(x,t). \tag{8}$$

E_F^n is the so-called Floquet quasi-energy and $\Phi_F^n(x,t)$ is a periodic function of time, with the same periodicity of the Hamiltonian, i.e.

$$\Phi_F^n(x,t+T) = \Phi_F^n(x,t) \qquad T = \frac{2\pi}{\omega}. \tag{9}$$

When using the BS space, the time-dependent Schrödinger equation becomes

$$\left[\omega_0\left(2z\frac{\mathrm{d}}{\mathrm{d}z}+1\right) + \frac{\lambda}{\sqrt{2\omega_0}}\left(\frac{\mathrm{d}}{\mathrm{d}z}+z\right)\cos\omega t\right]\Psi = \mathrm{i}\,\frac{\partial\Psi}{\partial t} \tag{10}$$

where $\Psi = \Psi(z,t)$. This equation is a first-order linear partial differential equation, involving an unknown function Ψ and two independent variables z and t, that can be written in the following way:

$$P\frac{\partial\Psi}{\partial z} + Q\frac{\partial\Psi}{\partial t} = R \tag{11}$$

where P, Q and R are functions of Ψ, z and t. Now, according to a method proposed by Lagrange (see, e.g., [16]), which states that the general solution will be $\phi(u,v) = 0$, provided $u(z,t,\Psi) = A$ and $v(z,t,\Psi) = B$, there are two independent solutions of the following auxiliary system (called the Lagrange system):

$$\frac{\mathrm{d}z}{P} = \frac{\mathrm{d}t}{Q} = \frac{\mathrm{d}\Psi}{R} \tag{12}$$

where A and B are integration constants. We also note that the functional form of ϕ is arbitrary.

Using the values of P, Q and R that result after writing the Lagrange system for equation (10), we take the first and second members of equation (12) as our first auxiliary equation

$$\frac{\mathrm{d}z}{2\omega_0 z + \frac{\lambda\cos\omega t}{\sqrt{2\omega_0}}} = \frac{\mathrm{d}t}{-\mathrm{i}}. \tag{13}$$

In order to solve this equation we make a change of variable $z(t) = M(t)\,\mathrm{e}^{2\mathrm{i}\omega_0 t}$. Although this may seem contradictory, since z and t are considered independent variables above, we stress that the basic assumption in the Lagrange method is that the independent variables appearing in equation (11) must be considered as dependent variables when solving the auxiliary system. Introduction of the change of variable in equation (13) leads to the following:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\lambda\mathrm{i}}{\sqrt{2\omega_0}}\,\mathrm{e}^{-2\mathrm{i}\omega_0 t}\cos\omega t \tag{14}$$

which after integration gives for $z(t)$ the value

$$z = \gamma^* + A\,\mathrm{e}^{2\mathrm{i}\omega_0 t} \tag{15}$$

where A is an arbitrary constant and γ is given by

$$\gamma(t) = \frac{\lambda}{\sqrt{2\omega_0}\left(\omega^2 - 4\omega_0^2\right)}\left\{2\omega_0\cos\omega t - \mathrm{i}\omega\sin\omega t\right\}. \tag{16}$$

The first solution of the Lagrange system can be written as

$$u(z,t) = (z-\gamma^*)\,\mathrm{e}^{-2\mathrm{i}\omega_0 t} = A. \tag{17}$$

Now, by making use of equation (12) for our second auxiliary equation

$$\frac{\mathrm{d}t}{-\mathrm{i}} = -\frac{\mathrm{d}\Psi}{\left(\omega_0 + \frac{\lambda z \cos\omega t}{\sqrt{2\omega_0}}\right)\Psi} \tag{18}$$

which can be integrated as an ordinary differential equation to obtain

$$-\mathrm{i}Et - \frac{\lambda\mathrm{i}}{2\omega\sqrt{2\omega_0}}\gamma^* \sin\omega t + A\gamma\,\mathrm{e}^{2\mathrm{i}\omega_0 t} - \ln\Psi = \ln L \tag{19}$$

where L is an integration constant and

$$E = \omega_0 + \frac{\lambda^2}{2\left(\omega^2 - 4\omega_0^2\right)}. \tag{20}$$

The solution of the second Lagrange equation can be written as

$$v(z, t, \Psi) = \mathrm{e}^{-\mathrm{i}Et}\,\mathrm{e}^{\beta+\gamma z}\Psi^{-1} = B \tag{21}$$

where B is a constant and

$$\beta(t) = \frac{-\mathrm{i}\lambda\sin\omega t}{2\omega\sqrt{2\omega_0}}\gamma(t) - \frac{\omega_0}{4}\left\{\frac{2\lambda}{\omega^2 - 4\omega_0^2}\right\}^2. \tag{22}$$

In order to proceed we need a specific functional form for the general solution $\phi(u, v)$. Based on the fact that the Floquet functions resulting in our approach become the harmonic oscillator functions when the field amplitude goes to zero, we propose for $\phi(u, v)$ the following form:

$$\phi(u, v) = u^{-n} - v \qquad (n \text{ positive integer}). \tag{23}$$

This choice of ϕ provides us with the Floquet functions $\Psi_F^n(z, t)$ in the BS space, with the Floquet quasi-energies given by

$$E_F^n = \omega_0(2n+1) + \frac{\lambda^2}{2\left(\omega^2 - 4\omega_0^2\right)} \tag{24}$$

and the corresponding functions $\Phi_F^n(z, t)$ are

$$\Phi_F^n(z, t) = \mathrm{e}^{\beta(t)}\,\mathrm{e}^{\gamma(t)z}(z - \gamma^*)^n. \tag{25}$$

These functions satisfy the condition of periodicity imposed by the Floquet theorem, since both γ and β have period T, as can be seen in equations (16) and (22). On the other hand, just as in the harmonic oscillator case, the quantization (n a positive integer) comes from the analyticity condition that the functions must satisfy (i.e. they must have derivatives everywhere except at $z = \infty$). It can also be seen that in the limit $\lambda \to 0$, both γ and β approach zero, so that the functions Φ_F^n approach z^n, the harmonic oscillator functions in the BS space, and the Floquet quasi-energies E_F^n approach $\omega_0(2n+1)$.

Finally, if we want to transform this solution to the configuration space, we use the displacement operator, so that

$$(z - \gamma^*)^n = \mathrm{e}^{-\gamma^*\frac{\mathrm{d}}{\mathrm{d}z}}z^n. \tag{26}$$

Bearing in mind that the harmonic oscillator eigenfunctions in the BS space are $\chi_n(z) = z^n$, and that the annihilation operator is $\frac{\mathrm{d}}{\mathrm{d}z}$, we can write Φ_F^n in the configuration space as

$$\Phi_F^n(x, t) = \mathrm{e}^{\beta(t)}\,\mathrm{e}^{\gamma(t)a^\dagger}\,\mathrm{e}^{-\gamma^*(t)a}\chi_n(x) \tag{27}$$

which agrees with the solutions previously obtained [5]. For the sake of completeness, we also note that this equation can be rearranged as

$$\Phi_F^n(x, t) = \mathrm{e}^{\delta(t)}\,\mathrm{e}^{\epsilon(t)(a+a^\dagger)}\,\mathrm{e}^{-\eta(t)(a-a^\dagger)}\chi_n(x) \tag{28}$$

where

$$\delta(t) = \frac{\mathrm{i}\lambda^2}{4\omega} \frac{4\omega_0^2 + \omega^2}{\left(\omega^2 - 4\omega_0^2\right)^2} \sin 2\omega t \tag{29}$$

$$\epsilon(t) = \frac{-\mathrm{i}\lambda\omega}{\sqrt{2\omega_0}\left(\omega^2 - 4\omega_0^2\right)} \sin \omega t \tag{30}$$

$$\eta(t) = \frac{\sqrt{2\omega_0}\lambda}{\omega^2 - 4\omega_0^2} \cos \omega t. \tag{31}$$

Using the definitions of $a, a^\dagger$ in terms of x and $\mathrm{d}/\mathrm{d}x$, and the displacement operator again, $\Phi_F^n(x,t)$ can be written as

$$\Phi_F^n(x,t) = \mathrm{e}^{\delta(t)} \, \mathrm{e}^{\sqrt{2\omega_0}\,\epsilon(t)x} \chi_n(x - x_c(t)) \tag{32}$$

where

$$x_c(t) = \frac{2\lambda}{\omega^2 - 4\omega_0^2} \cos \omega t. \tag{33}$$

This form coincides with the solution given by Breuer and Holthaus [15].

4. Conclusions

We have presented a new and simple method to obtain the Floquet quasi-energies and functions for a harmonic oscillator with an external monochromatic field. The simplicity results from working in the BS space, where the wave equation becomes a first-order partial differential equation as compared with the usual second-order Schrödinger equation. This method compares favourably with a previous one [5] based on the Lie algebraic techniques, the advantage being that the present approach is almost straightforward at least for the case presented here. Although Husimi and the path integration methods are more transparent from the physical point of view, the mathematical details are more cumbersome than the method presented here. However, the application to some other cases is in progress. A study of the relationship between the path integration method and the actual analysis is also under way.

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